# Measuring the parity of an $N$-qubit state

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We present a scheme for a projective measurement of the parity operator $P_z = \prod_{i=1}^N \sigma_z^{(i)}$ of $N$-qubits. Our protocol uses a single ancillary qubit, or a probe qubit, and involves manipulations of the total spin of the $N$ qubits without requiring individual addressing. We illustrate our protocol in terms of an experimental implementation with atomic ions in a two-zone linear Paul trap, and further discuss its extensions to several more general cases.

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Quantum measurement, the only approach to gain information on the state of a quantum system, is an essential ingredient of the quantum theory [1, 2, 3, 4]. In quantum computation and information, systems of many qubits are prepared, processed, and measured. Quantum measurements on a system of many qubits can be simply classified into single qubit measurements and quantum collective measurements, or measurements on many qubits as a whole at the same time. A typical example of quantum collective measurement is the Bell state measurement on two qubits as in quantum teleportation, where the four entangled and orthogonal Bell states have to be distinguished [5, 6, 7]. Although quantum collective measurements are more difficult to implement in laboratory systems than single qubit measurements, they are a necessary element for useful applications of quantum computations, especially for the implementation of quantum error corrections. The theoretical foundation of quantum collective measurements is well established [3, 8]. In principle, they can be implemented by a series of unitary transformations and single qubit measurements with the aid of auxiliary (ancillary) qubits [8]. For specific laboratory systems, however, how to implement a given quantum collective measurement *efficiently* remains a challenging task.

In this article, we suggest a scheme for a collective measurement of the parity operator $P_z$ of $N$ qubits. The efficiency of our protocol is due to: 1) only one ancillary qubit (probe qubit) is introduced; and 2) its overall complexity does not explicitly depend on the number of qubits $N$.

The parity operator $P_z \equiv \prod_{i=1}^N \sigma_z^{(i)}$ of $N$ qubits is important for quantum information science. It is frequently invoked in the foundations of quantum mechanics for its direct connections to nonlocal quantum correlations. Its measurement is also important for highly precise sensing schemes [9, 10, 11], e.g. in the realization of Heisenberg limit atomic metrology with a maximally entangled state of $N$ qubits. The simplest scheme for measuring $P_z$ involves projective measurements on individual qubits ($|0\rangle$ or $|1\rangle$), destroying coherence among multi-qubit states within the same parity subspace along the way. An instructive method was proposed recently by Fenner *et al.* [12], which uses pairwise Heisenberg interaction and $N+1$ auxiliary qubits.

We start with an intuitive method for measuring $P_z$ as described in the following steps: (i) We introduce a probe qubit (denoted by 0) prepared initially in state $|0^z\rangle_0$, the eigenstate of $\sigma_z^{(0)}$ with eigenvalue $+1$; (ii) We perform successively $N$ control-NOT (C-NOT) gates between the i-th qubit (control) and the probe qubit, where $i$ runs from 1 to $N$; (iii) We measure the probe qubit in the basis states of $\sigma_z^{(0)}$. If the measurement result is $+1$, the parity of the $N$ qubit state is $+1$, and the $N$ qubit system is projected into the subspace with $+1$ parity; Otherwise the parity is $-1$, and the state of $N$ qubits is projected into the subspace with $-1$ parity. The complexity of such a scheme comes mainly from step (ii), where the probe qubit is required to interact sequentially with qubit-1 to qubit-$N$, to effect the $N$ C-NOT gates.

The scheme we propose is based on the following idea. We intend to perform the $N$ C-NOT gates not sequentially, but in only one step. In fact, we execute $N$ two-qubit phase gates [see Eq. (3)] instead because it is more convenient to scale them up due to the invariance under interchange of the two qubits. Our protocol is conveniently illustrated in terms of the graph as in Fig. 1. In the first step, we perform phase gates to every qubit pairs from the $N+1$ qubits, i.e. the probe qubit plus the $N$ system qubits. In the second step we perform phase gates to every qubit pairs of the $N$ system qubits. In Fig. 1, a black solid dot denotes a qubit (for $N=5$). A phase gate between two qubits is represented by a line that connects the two dots. In the first step, a total of $C_{N+1}^2$ lines are needed; while $C_N^2$ lines result from the second step. Because the inverse of a phase gate is itself, the net effect of the above two steps is a graph with $N(=C_{N+1}^2 - C_N^2)$ solid lines connecting the ancillary probe qubit and the $N$ system qubits as in Fig. 1. The dashed lines denote the actions of repeated phase gates, or null events.

We now describe the four steps of our protocol:

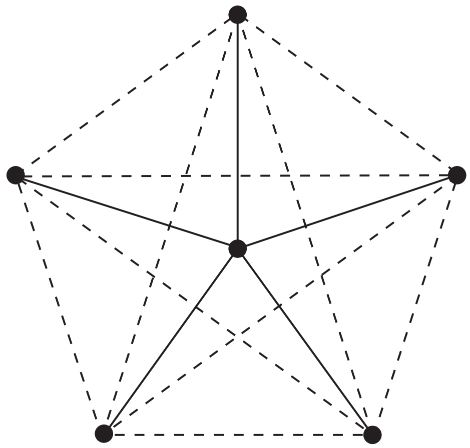

FIG. 1: A graph illustrating our protocol for measuring the parity of a $N = 5$ qubit state. The ancillary probe qubit is located at the center, and the parity of the five surrounding system qubits is to be measured.

(i) The probe qubit is initialized to

$$|0^x\rangle_0 = \frac{|0^z\rangle_0 + |1^z\rangle_0}{\sqrt{2}}. \quad (1)$$

(ii) Phase gates are applied to all pairs of qubits, the probe qubit and the $N$ system qubits, through the unitary transformation

$$U^{(0,1,\cdots,N)}(N+1, J_z) = \prod_{(ij)} U^{(ij)}, \;\; i,j \in \{0,1,2,\cdots,N\} \quad (2)$$

where $(ij)$ denotes the $i$-th and $j$-th qubit pair, on which the action of the phase gate $U^{(ij)}$ is defined as

$$U^{(ij)}|a^z\rangle_i|b^z\rangle_j = (-1)^{ab}|a^z\rangle_i|b^z\rangle_j, \;\; a,b \in \{0,1\}. \quad (3)$$

Several earlier theoretical studies have shown that the unitary transformation $U^{(0,1,\cdots,N)}$ can be realized (in one step) by the following Hamiltonian [13, 14],

$$H(N+1, J_z) = \chi\left(\frac{J_z^2}{2} - \frac{N}{2}J_z + \frac{N^2-1}{8}\right), \quad (4)$$

with $J_z \equiv \sum_{i=0}^{N} \sigma_z^{(i)}/2$ the $z$ component of the total spin of the $N+1$ qubits. In fact, this is easy to see as

$$U^{(0,1,\cdots,N)}(N+1, J_z) = e^{-iH(N+1,J_z)\pi/\chi}.$$

(iii) The unitary transformation $U^{(1,2,\cdots,N)}(N, J_z)$ is applied to the N system qubits.

(iv) The measurement of $\sigma_x^{(0)}$ on the probe qubit is carried out. If the result is $+1$ (collapsing the probe to state $|0^x\rangle_0$), the parity of the $N$ qubit state is $+1$, and the state of the $N$ system qubits is projected into the subspace of parity $+1$; Otherwise the parity is $-1$, and the state of $N$ qubits is projected into the subspace of parity $-1$.

Because different phase gates $U^{(ij)}$s commute with each other, the step (iii) of the above protocol can be moved to either before the step (ii) or after the step (iv). The unitary transformation $U^{(1,2,\cdots,N)}(N, J_z)$ is also known to be capable of generating specific cluster states, e.g. an $N$-qubit GHZ state, when the $N$-qubit state is initialized to a product state $\prod_{i=1}^{N} |0^x\rangle_i$ [15, 16]. Certain error-correcting codes can be realized with cluster states, or more generally with graph states [17]. As mentioned before, the Hamiltonian (4) can be realized in various two state systems, e.g., quantum degenerate atoms [14], trapped atomic ions [13], cavity QED systems [18], and solid state Josephson junctions [19, 20].

We now prove that our protocol indeed corresponds to a projective measurement of $P_z$ for an $N$-qubit system in a pure state

$$|\psi\rangle_{12\cdots N} = \sum_{a_1,a_2,\cdots,a_N=0}^{1} \left(c_{a_1a_2\cdots a_N} \prod_{i=1}^{N} |{a_i}^z\rangle_i\right). \quad (5)$$

From the definition of the two qubit phase gate (3), it is easy to find that

$$\begin{aligned} U^{(0;1,2,\cdots,N)} &\equiv U^{(1,2,\cdots,N)}(N, J_z) U^{(0,1,\cdots,N)}(N+1, J_z) \\ &= \prod_{i=1}^{N} U^{(0i)}. \end{aligned} \quad (6)$$

Then the state of the whole system before the step (iv) is simply

$$\begin{aligned} &U^{(0;1,2,\cdots,N)}|0^x\rangle_0|\psi\rangle_{12\cdots N} \\ &= |0^x\rangle_0|\psi^e\rangle_{12\cdots N} + |1^x\rangle_0|\psi^o\rangle_{12\cdots N}, \end{aligned} \quad (7)$$

where

$$|1^x\rangle_0 = \frac{|0^z\rangle_0 - |1^z\rangle_0}{\sqrt{2}}, \quad (8)$$

and

$$|\psi^e\rangle_{12\cdots N} = \frac{1+P_z}{2}|\psi\rangle_{12\cdots N} = \delta_{P_z,+1}|\psi\rangle_{12\cdots N}, \quad (9)$$

$$|\psi^o\rangle_{12\cdots N} = \frac{1-P_z}{2}|\psi\rangle_{12\cdots N} = \delta_{P_z,-1}|\psi\rangle_{12\cdots N}, \quad (10)$$

are respectively the even and odd parity parts of the state (5). In deriving Eqs. (7), (9), and (10), we have made use of the eigen-equation for the parity operator

$$P_z \prod_{i=1}^{N} |{a_i}^z\rangle_i = (-1)^{\sum_{i=1}^{N} a_i} \prod_{i=1}^{N} |{a_i}^z\rangle_i. \quad (11)$$

Through the steps (ii) and (iii) we entangle the eigenstates of $\sigma_x^{(0)}$ for the probe qubit with the parity eigenstates of the $N$-qubit system, as illustrated by solid lines

in Fig. 1. A measurement of $\sigma_x^{(0)}$ on the probe qubit thus leads to a projective measurement of the parity operator $P_z$ on the $N$ system qubits.

Similarly, the probe qubit can be initialized to other states, e.g. the $|1^x\rangle_0$. We then find analogously

$$U^{(0;1,2,\cdots,N)}|1^x\rangle_0|\psi\rangle_{12\cdots N} = |0^x\rangle_0|\psi^o\rangle_{12\cdots N} + |1^x\rangle_0|\psi^e\rangle_{12\cdots N}, \qquad (12)$$

i.e. entanglement is again established, and the parity of the $N$ system qubits is revealed from measuring the probe qubit. Clearly, our protocol and the above proof also apply to mixed states of $N$ qubits.

Compared to the intuitive scheme first discussed, our protocol possesses two main advantages: (1) The Hamiltonian $H(N, J_z)$ is a function of the total qubit number $N$ and the $z$ component of the total spin, thus addressing of individual qubits is not required, or explicitly, the complexity of our protocol is independent of $N$; (2) The ancillary probe qubit has the same interaction properties as the $N$ system qubits, so one type of qubits can be used for experimental implementations. We can simply consider one of the $N+1$ qubits as the probe.

We now contrast our protocol with other schemes as well as adopted experimental approaches for collective measurements of parity. The eigenvalues for the parity $P_z$ of $N$ qubits take two alternative values, $+1$ or $-1$. Their corresponding subspaces are thus $2^{N-1}$ fold degenerate, and can be further characterized by the number of 1s, or $N^{(-)} = \sum_{i=1}^{N}(1-\sigma_z^{(i)})/2$. Our projective measurement protocol collapses the N-qubit state into the degenerate subspace of a fixed parity, but maintains quantum coherence between states of different $N^{(-)}$ because $\left[N^{(-)}, P_z\right] = 0$. Previous schemes [9, 11] and approaches adopted by several recent experiments [10, 21, 22], on the other hand, measure $P_z$ by counting the number of qubits in state $|1^z\rangle$, or evaluating $N^{(-)}$ from results of individual qubit measurements. We note that $P_z$ is $+1$ (or $-1$) when $N^{(-)}$ is even (or odd). In these schemes of measuring $P_z$ through $N^{(-)}$, coherence between states with different $N^{(-)}$ but with the same parity $P_z$ is completely destroyed, thus they cannot be used for quantum error correction.

For a simple implementation of our protocol, we consider atomic ions in a two-zone radio frequency linear Paul trap [23]. The ion configurations during each of the above four steps are illustrated in Fig. 2. In steps (i), (iii), and (iv), the ancillary probe ion is in the first zone, while the $N$ system ions are in the second zone; In step (ii), all the $N+1$ ions are in the second zone, where the unitary transformations $e^{i\chi J_x^2}$ and $e^{i(\alpha J_x+\beta J_y)}$ can be effected experimentally [23]. Thus one can use the equalities

$$e^{i\chi J_z^2} = e^{i\frac{\pi}{2}J_y}e^{i\chi J_x^2}e^{-i\frac{\pi}{2}J_y}, \qquad (13)$$
$$e^{i\alpha J_z} = e^{i\frac{\pi}{2}J_y}e^{i\alpha J_x}e^{-i\frac{\pi}{2}J_y}, \qquad (14)$$

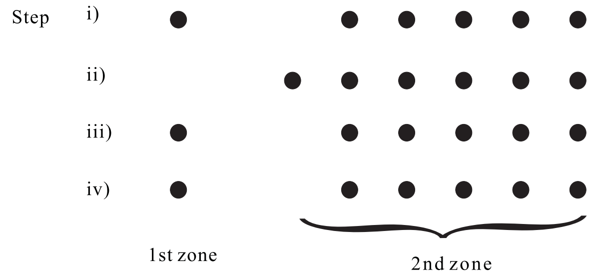


FIG. 2: From top to bottom, the configurations of ions in each of the four steps for our protocol implemented in a two-zone radio frequency linear Paul trap.

to realize the required unitary transformations (2) in steps (ii) and (iii). It seems such capabilities already exist in current experimental systems [23]. As already pointed out, steps in the second zone only involve the manipulation of the total spin, no individual ion qubit addressing is required.

In the following, we extend our protocol to more general quantum collective measurements. For instance, let's consider the measurement of a general element of the Pauli group $G_N$

$$\mathcal{P} = P_x^{(1)} \otimes P_y^{(2)} \otimes P_z^{(3)} \otimes P_e^{(4)}, \qquad (15)$$

where $P_\mu^{(\alpha)} = \prod_{i=1}^{N_\alpha}\sigma_\mu^{(\alpha,i)}$, ($\alpha = 1,2,3,4$; and $\mu = x,y,z,e$), with $\sigma_e$ the identity operator in a two-dimensional Hilbert space, and the total qubit number $N = N_1 + N_2 + N_3 + N_4$. This extension is not only necessary for the completeness of our theory, it is also required for implementing quantum error corrections. Our strategy for quantum collective measurement of $\mathcal{P}$ now proceeds as follows. First, the $N$ qubits are grouped into four sets of $N_1$, $N_2$, $N_3$, and $N_4$ qubits as in (15). With only one ancillary probe qubit (0), $\mathcal{P}$ is measured in the end after successive applications of steps i), ii), and iii) of our protocols to $P_x^{(1)}$, $P_y^{(2)}$, and $P_z^{(3)}$. The probe qubit measurement thus makes this scheme a projective measurement protocol for $\mathcal{P}$.

To accomplish the above strategy for measuring $\mathcal{P}$, we extend our protocol for the parity operator $P_z$ to collective operators $P_x$ and $P_y$, by making use of state rotations, or equivalently cyclic coordinate transformations $x \to y \to z \to x$. Such transformations enable measuring $P_x$ and $P_y$ in similar fashions to the protocol for measuring $P_z$ as discussed above. Explicitly, one first prepares the probe qubit in the state $|0^y\rangle_0$, and then effects the unitary transformations needed for the measurement of $P_x$; Next a unitary transformation for the probe qubit is carried out: $\{|0^y\rangle_0 \to |0^z\rangle_0,\ |1^y\rangle_0 \to |1^z\rangle_0\}$, and $P_y$ can be measured accordingly with appropriate unitary transformations; One then performs a final unitary transformation on the probe qubit: $\{|0^z\rangle_0 \to |0^x\rangle_0,\ i|1^z\rangle_0 \to |1^x\rangle_0\}$, and measures the parity operator $P_z$ using our

protocol; In the end, a projective measurement of $\sigma_x^{(0)}$ for the probe qubit reveals the value of $\mathcal{P}$.

Before conclusion, we briefly generalize our protocol to the case of $d$-level systems (qudits). A natural generalization of two state Pauli matrices $\sigma_z$ and $\sigma_x$ are operators $Z$ and $X$, satisfying $XZ = qZX$ with $q \equiv \exp(i2\pi/d)$ [24, 25, 26, 27]. The parity of $N$ qudits can then be defined as $P_z \equiv \prod_{i=1}^{N} Z^{(i)}$, whose eigenvalues take on values $q^i$, $(i = 0, 1, \cdots, d-1)$. If we define the appropriate two-qudit phase gate as

$$U^{(ij)}|a^z\rangle_i|b^z\rangle_j = q^{ab}|a^z\rangle_i|b^z\rangle_j, \quad a, b \in \{0, 1, \cdots, d-1\}, \tag{16}$$

our above protocol for qubits can be directly generalized to qudits. For example, Equation (7) now becomes

$$U^{(0;1,2,\cdots,N)}|0^x\rangle_0|\psi\rangle_{12\cdots N} = \sum_{n=0}^{d-1} \delta_{P_z,q^n}|n^x\rangle_0|\psi\rangle_{12\cdots N}. \tag{17}$$

The measurement of $X^{(0)}$ therefore gives rise to a projective measurement of $P_z$ for the $N$ qudits.

In summary, we have proposed a theoretical protocol for a projective measurement of the parity $P_z$ of an $N$-qubit state, making use of a moving probe qubit and collective spin interactions. Provided symmetric spin interactions can be engineered without addressing individual spins, the complexity of our protocol becomes independent of the qubit number $N$. We have suggested an experimental implementation of our protocol with ions in a two-zone linear Paul trap. It seems that all necessary steps are within the limits of current laboratory capabilities [23]. Compared to previous schemes for measuring the same parity operator, our protocol makes use of nonlinear interactions to effect a quantum collective measurement in terms of one ancillary qubit, rather than individual projective measurements on all qubits. Thus, our projective scheme maintains the quantum coherence in the subspace of the parity of $N$ qubits, and can be applied to quantum error correction. Finally, we have generalized our protocol to quantum collective measurements of any elements of the Pauli group for $N$ qubits, as well as to systems of qudits.

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